Category Theory Notes

Edward Burkard

6. Categories and Sheaves

6.1. **The Language of Categories.** Intuitively, a category can be thought of as a bunch of "dots and arrows", where the arrows satisfy some certain properties. The "dots" are the objects of a category, and the "arrows" are the morphisms. Formally, we define a category as:

Definition 1 (Category)**.** *A* category $\mathcal{C}$ *consists of:*

- *a class of* objects*, denoted* $Ob(\mathcal{C})$
- *for any* $A, B, C \in Ob(\mathcal{C})$*, a class of* morphisms*,* $Hom_{\mathcal{C}}(A,B)$ *, together with a composition*

$$\begin{array}{rrcl} \circ: & Hom_{\mathcal{C}}(A,B) \times Hom_{\mathcal{C}}(B,C) & \to & Hom_{\mathcal{C}}(A,C) \\ & (f,g) & \mapsto & g \circ f \end{array}$$

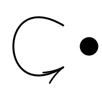
  which is
  1. *associative*
  2. *has an identity: for* $A, B \in Ob(\mathcal{C})$ *there is* $id_A \in Hom_{\mathcal{C}}(A,A)$ *and* $id_B \in Hom_{\mathcal{C}}(B,B)$ *such that for all* $f \in Hom_{\mathcal{C}}(A,B)$ *we have* $id_B \circ f = f = f \circ id_A$*.*

We call a category small *if* $Ob(\mathcal{C})$ *is a set and all* $Hom_{\mathcal{C}}(A,B)$ *are sets. We say* $\mathcal{C}$ *is* locally small *if only the second condition is satisfied.*

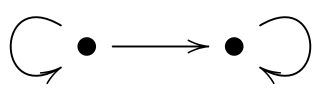
Here are some trivial examples of categories:

0: This is the category with no objects and no morphisms. This is it pictorally:

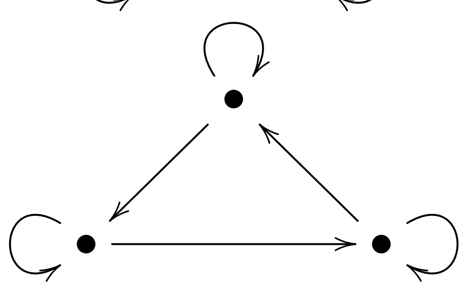
1: This is the category with one object and one morphism. The definition of a category forces this morphism to be the identity morphism on the one object. Here it is pictorally:



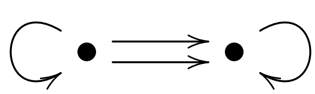
2: This is the category with two objects and one morphism between them (of course me must still have the identity morphisms on each element). Pictorally **2** looks like:



3: I will just draw this one:



$\downarrow\downarrow$: Again, I will just draw this one:



Now for some nontrivial examples:

Category	Objects	Morphisms
Set	Sets	Functions
Top	Topological Spaces	Continuous Maps
Man	Smooth Manifolds	Smooth Maps
$\mathbf{Vect}_M$	Smooth Vector Bundles over M	Smooth Maps which are linear of fibers
$\mathbf{Open}(X)$	Open subsets of a topological space X	Given $U, V \in \mathrm{Ob}\,(\mathbf{Open}(X))$ $\mathrm{Hom}_{\mathbf{Open}(X)}\,(U,V) = \begin{cases} \{*\} & U \subset V \\ \varnothing & otherwise \end{cases}$
Grp	Groups	Group Homomorphisms
Ab	Abelian Groups	Group Homomorphisms
R-**Mod**	Left R-modules	R-module Homomorphisms
k-**Vect**	k-Vector Spaces	k-linear maps
Pos	Partially Ordered Sets (Posets)	Order Preserving Maps
$\mathbf{Ord(}P\mathbf{)}$	Elements of a Poset P	Given $x, y \in \mathrm{Ob}\,(\mathbf{Ord}(P))$ $\mathrm{Hom}_{\mathbf{Ord}(P)}\,(x,y) = \begin{cases} \{*\} & x \leq y \\ \varnothing & otherwise \end{cases}$
$\mathbf{Ord}(P)$	$\{*\}$	$\mathrm{Hom}_{\mathbf{Ord}(P)}\,(*,*) = G$ (composition coincides with group multiplication)
Δ (Simplicial)	$\{[i] \mid i \geq -1\}$ $[i] = \{0, ..., i\}$ for $i > -1$ $[-1] = \varnothing$	Order Preserving Maps

Naturally, there is a concept of a *subcategory*. Here are some examples of subcategories from the list above:

(1) **Top** is a subcategory of **Set**

(2) **Grp** is a subcategory of **Set**
(3) **Ab** is a subcateory of **Grp**

Sometimes the Hom-sets of the subcategory are the same as in the whole category. In this case, we call the subcategory a *full subcategory.* To state this formally:

Definition 2 (Full Subcategory)**.** *Let $\mathcal{D}$ be a subcategory of $\mathcal{C}$. We say $\mathcal{D}$ is a* full subcategory *of $\mathcal{C}$ if for all $A, B \in Ob(\mathcal{D})$*

$$Hom_{\mathcal{D}}(A,B) = Hom_{\mathcal{C}}(A,B)\ .$$

An example of a full subcategory is the third one above, and non examples are the other two.

Given a category, we may obtain new categories in various ways. One such way is to "turn around all the arrows". A more precise definition is:

Definition 3 (Opposite of a Category)**.** *Give a category $\mathcal{C}$, we define its* opposite*, denoted $\mathcal{C}^{op}$, as the category with the same objects as $\mathcal{C}$, but with $Hom_{\mathcal{C}^{op}}(A,B) := Hom_{\mathcal{C}}(B,A)$. Composition in this category is defined as follows: if $f^* \in Hom_{\mathcal{C}^{op}}(A,B)$ comes from $f \in Hom_{\mathcal{C}}(B,A)$, then composition with $g^* \in Hom_{\mathcal{C}^{op}}(B,C)$ is given by*

$$g^* \circ f^* = (f \circ g)^*.$$

A nice example of an opposite category is the opposite of k-**vect**. We get an equivalence of categories (to be defined) between k-**vect** and $k - \mathbf{vect}^{op}$ by dualization, i.e.

$$\begin{array}{rcl} k - \mathbf{vect} & \xrightarrow{*} & k - \mathbf{vect}^{op} \\ V & \mapsto & V^* \\ \phi : V \to W & \mapsto & \phi^* : W^* \to V^* \end{array}$$

Of course things don't always work out so nicely. Consider, for example, the category **Set**. The best we can do here (and this is probably still nicer than what we can do in most cases) is embed $\mathbf{Set}^{op}$ into **Set** via

$$\begin{array}{rcl} \mathbf{Set}^{op} & \longrightarrow & \mathbf{Set} \\ A & \mapsto & \wp(A) \\ f^* : A \to B & \mapsto & f^{-1} : \wp(A) \to \wp(B) \end{array}$$

Assuming that the category $\mathcal{C}$ is not too large, one more construction we can do is to quotient by isomorphisms. The quotient category, call it $\mathcal{C}'$ has objects a representative from each equivalence class of objects (chosen by the axiom of choice) (two objects A and B in $\mathcal{C}$ are isomorphic if an element of $\mathrm{Hom}_{\mathcal{C}}(A,B)$ is an isomorphism in the category $\mathcal{C}$). $\mathcal{C}'$ is the subcategory of $\mathcal{C}$ generated by these objects.

Now we will focus on special types of objects in a category (which will become important in the next section).

Definition 4 (Initial/Terminal Object)**.** *Let $\mathcal{C}$ be a category. An object I in $\mathcal{C}$ is called* initial *if $Hom_{\mathcal{C}}(I,A)$ has one element for all objects A in $\mathcal{C}$. Analogously, a* terminal *object T in $\mathcal{C}$ is one such that $Hom_{\mathcal{C}}(A,T)$ consists of one element for all A. An object which is both initial and terminal is called a* zero *or* null *object.*

Not every category has initial or terminal objects (or either). Here are some examples of categories and their initial and/or terminal objects:

Category	Initial Object	Terminal Object
Set	$\varnothing$	$\{*\}$
Grp	$\{e\}$	$\{e\}$
R-**Mod**	$\{0\}$	$\{0\}$
Top	$\varnothing$	$\{*\}$
$\mathbf{Set}_{\neq\varnothing}$	none	$\{e\}$
Field	none	none
$\mathbf{Ord}(P)$	The least element (if it exists)	The greatest element (if it exists)

Now we turn our attention to morphisms between categories.

Definition 5 (Functor)**.** *Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A* functor*, $F : \mathcal{C} \to \mathcal{D}$, is a pair of maps*

$$\begin{array}{rrcl} F: & Ob(\mathcal{C}) & \to & Ob(\mathcal{D}) \\ F_{A,B}: & Hom_{\mathcal{C}}(A,B) & \to & Hom_{\mathcal{D}}(F(A),F(B)) \end{array}$$

such that:

- $F(id_A) = id_{F(A)}$
- $F(f \circ g) = F(f) \circ F(g)$

(If instead $F_{A,B} : \mathrm{Hom}_{\mathcal{C}}(A,B) \to \mathrm{Hom}_{\mathcal{D}}(F(B),F(A))$ and the second bullet in the definition is replaced with $F(f \circ g) = F(g) \circ F(f)$, we call F a *contravariant* functor. We can always turn a contravariant functor into a (covariant) functor by instead considering F from $\mathcal{C}^{op}$ to $\mathcal{D}$ instead.)

We call a functor *forgetful* if we lose structure in passing to the image. Here are some examples of functors:

(1) $\pi_n : \mathbf{Top}_* \to \mathbf{Grp}$
(2) $H^k : \mathbf{Top} \to \mathbf{Ab}$ (this is actually contravariant)
(3) $F : \mathbf{Grp} \to \mathbf{Set}$ (an example of a forgetful functor)

Now we have to ask what does it mean for two categories to be equivalent:

Definition 6 (Fully Faithful & Equivalence)**.** *A functor $F : \mathcal{C} \to \mathcal{D}$ is called* fully faithful *if for any A and B objects in $\mathcal{C}$ the map*

$$F_{A,B} : Hom_{\mathcal{C}}(A,B) \to Hom_{\mathcal{D}}(F(A),F(B))$$

is bijective. Moreover, we say F is an equivalence *of categories if, in addition, for any $D \in Ob(\mathcal{D})$ there is a $C \in Ob(\mathcal{C})$ such that $F(C)$ is isomorphic to D.*

Note that an equivalence of categories does not mean that there is a bijection on the classes of objects, but rather on isomorphism classes of objects. There is a notion of an isomorphism of categories which is as follows: We say that a functor $F : \mathcal{C} \to \mathcal{D}$ is an *isomorphism* of categories if there exists another functor $G : \mathcal{D} \to \mathcal{C}$ such that $G \circ F$ is the identity functor on $\mathcal{C}$ and $F \circ G$ is the identity functor on $\mathcal{D}$. An equivalence of categories also has a similar definition, in which the compositions are only required to be naturally isomorphic to the identity in a sense we are about to define. In the meantime, here is a simple exercise

Exercise 1.

(a) *Supply the details for the definition of the category of categories,* **Cat**. Hint: *You might have to restrict which kind of categories you consider as objects of this category, otherwise you will run into a Russel's paradox.*
(b) *Does* **Cat** *have an initial and/or terminal object? If so, identify them. Does* **Cat** *have a zero object?*
(c) *Is* **Cat** *a small category?*
(d) *If you are ambitious enough, make sense of* **Cat** *as a 2-category (I will not define this here, go look it up). You will need the following definition to do this.*

We also have a notion of a map between functors, known as a natural transformation.

Definition 7 (Natural Transformation)**.** *Let $\mathcal{C}$ and $\mathcal{D}$ be categories and let F and G be two functors from $\mathcal{C}$ to $\mathcal{D}$. A* natural transformation *is a function $\eta : F \xrightarrow{\bullet} G$ such that for all $C, C' \in Ob(\mathcal{C})$ there are morphisms $\eta_C : F(C) \to G(C)$ such that for all $f \in Hom_{\mathcal{C}}(C, C')$ the diagram*

$$\begin{tikzcd} F(C) \arrow[r, "F(f)"] \arrow[d, "\eta_C"'] & F(C') \arrow[d, "\eta_{C'}"] \\ G(C) \arrow[r, "G(f)"] & G(C') \end{tikzcd}$$

commutes.

A last definition in this section is one of types of morphisms in a category.

Definition 8 (Monomorphism/Epimorphism/Isomorphism)**.** *Let $\mathcal{C}$ be a category and $A, B, C \in Ob(\mathcal{C})$. A morphism $f \in Hom_{\mathcal{C}}(B, C)$ is called a* monomorphism *if for all $g, h \in Hom_{\mathcal{C}}(A, B)$ with $f \circ g = f \circ h$, we have $g = h$. $f \in Hom_{\mathcal{C}}(A, B)$ is called an* epimorphism *if for all $g, h \in Hom_{\mathcal{C}}(B, C)$ with $g \circ f = h \circ f$ we have $g = h$. $f \in Hom_{\mathcal{C}}(A, B)$ is an* isomorphism *is there is a $g \in Hom_{\mathcal{C}}(B, A)$ such that $f \circ g = id_B$ and $g \circ f = id_A$.*

6.2. **Additive and Abelian Categories.** In this section we will set up machinery that will be useful in later sections. This machinery consists of types of inherent operations in a category.

Definition 9 (Products/Coproducts/Biproducts). *Let $\mathcal{A}$ be a category and let $\{A_i \mid i \in I\}$ be a collection of objects in $\mathcal{A}$.*

(1) *A* product *for the family $\{A_i \mid i \in I\}$ is an object P (often denoted $\prod_i A_i$) of $\mathcal{A}$ together with a family of morphisms $\{\pi_i : P \to A_i \mid i \in I\}$ such that for any object Q and collection of morphisms $\{\phi_i : Q \to A_i \mid i \in I\}$, there is a unique morphism $\psi : Q \to P$ such that $\pi_i \circ \psi = \phi_i$. For $I = \{1, 2\}$, this pictorally looks like:*

$$\begin{tikzcd} & Q \arrow[dl, "\phi_1"'] \arrow[d, dashed, "\exists!\psi"] \arrow[dr, "\phi_2"] & \\ A_1 & A_1 \times A_2 \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & A_2 \end{tikzcd}$$

(2) *A* coproduct *for the family $\{A_i \mid i \in I\}$ is an object C (often denoted $\sum_i A_i$) of $\mathcal{A}$ together with a family of morphisms $\{\iota_i : C \to A_i \mid i \in I\}$ such that for any object D and collection of morphisms $\{\phi_i : Q \to A_i \mid i \in I\}$, there is a unique morphism $\psi : C \to D$ such that $\psi \circ \iota_i = \phi_i$. For $I = \{1, 2\}$, this pictorally looks like:*

$$\begin{tikzcd} A_1 \arrow[r, "\iota_1"] \arrow[dr, "\phi_1"'] & A_1 + A_2 \arrow[d, dashed, "\exists!\psi"] & A_2 \arrow[l, "\iota_2"'] \arrow[dl, "\phi_2"] \\ & D & \end{tikzcd}$$

(3) *Suppose now that $\mathcal{A}$ has a zero object. A* biproduct *for the family $\{A_i \mid i = 1, ..., n\}$ is an object B (often denoted $\oplus_i A_i$) of $\mathcal{A}$ which is both the product and coproduct of the family, and for which the collection of morphisms π_i and ι_i satisfy*

$$\pi_i \circ \iota_j = \begin{cases} id_{A_i}, & i = j \\ 0, & i \neq j \end{cases}.$$

For example, when $n = 2$, the "biproduct diagram" takes the form

$$\begin{tikzcd} A_1 \arrow[r, shift left, "\iota_1"] & A_1 \oplus A_2 \arrow[l, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] & A_2 \arrow[l, shift right, "\iota_2"'] \end{tikzcd}$$

$\mathcal{A}$ is said to have finite *products/coproducts/biproducts if it has products/coproducts/biproducts when I is a finite set.*

Now another definition

Definition 10 (Additive Category). *A category $\mathcal{A}$ is called* additive *if*

(1) *$\mathcal{A}$ has a zero object, 0, (This give a unique zero map between any two objects A and B via the unique $A \to 0 \to B$.)*
(2) *$Hom_{\mathcal{A}}(A, B)$ is an abelian gorup for all $A, B \in Ob(\mathcal{A})$ where the zero map is the identity and composition is bilinear,*
(3) *$\mathcal{A}$ has finite biproducts.*

Now we will construct the other type of category in the title, but first we need to define a few more things:

Definition 11 (Kernel/Cokernel and Image/Coimage)**.** *Let $\mathcal{A}$ be a category with zero morphisms and let $A, B \in Ob(\mathcal{A})$. Let $f \in Hom_{\mathcal{A}}(A, B)$.*

(1) *The* kernel *of f is a pair (K, k) where $K \in Ob(\mathcal{A})$ and $k : K \to A$ is such that $f \circ k = 0$ and if there is a $g \in Hom_{\mathcal{A}}(P, A)$ such that $f \circ g = 0$, there is a unique $h \in Hom_{\mathcal{A}}(P, K)$ such that $g = k \circ h$. That is*

$$\begin{tikzcd} K \arrow[r, "k"] & A \arrow[r, "f"] & B \\ & P \arrow[ul, "\exists! h"] \arrow[u, "g"'] & \end{tikzcd}$$

(2) *The* cokernel *of f is a pair (C, c) where $C \in Ob(\mathcal{A})$ and $c : B \to C$ is such that $c \circ f = 0$ and if there is a $q \in Hom_{\mathcal{A}}(B, Q)$ such that $q \circ f = 0$, there is a unique $d \in Hom_{\mathcal{A}}(C, Q)$ such that $d \circ c = q$. That is*

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, "c"] \arrow[d, "q"'] & C \arrow[dl, "\exists! d"] \\ & Q & \end{tikzcd}$$

(3) *The* image *of f is the kernel of its cokernel, and the* coimage *of f is the cokernel of its kernel.*

It is worth noting that kernels are always monomorphisms and cokernels are always epimorphisms (see Exercise 2 of section 6.2 in the notes). Finally, we have the technology to define an abelian category:

Definition 12 (Abelian Category)**.** *An* abelian category *is an additive category such that*

(1) *every morphism has a kernel and cokernel*
(2) *the natural map, σ, from the coimage of a morphism to the image of the morphism is an isomorphism*

equivalently, we can phrase (2) as:

(2') *every morphism $f : A \to B$ has a factorization*

$$\begin{tikzcd} \ker(f) \arrow[r, "k"] & A \arrow[r, "u"] \arrow[rr, bend left, "f"] & im(f) \arrow[r, "v"] & B \arrow[r, "c"] & coker(f) \end{tikzcd}$$

where u and v are the natural maps. u is an epimorphism and v is a monomorphism (see Exercise 2 of section 6.2 of the notes).

What is this natural map in (2)? To construct it, consider the diagram

$$\begin{tikzcd}
\ker(f) \arrow[r, "k"] & A \arrow[rr, "f"] \arrow[d, "u"'] & & B \arrow[r, "c"] & \operatorname{coker}(f) \\
 & \operatorname{coim}(f) = \operatorname{coker}(k) \arrow[urr, dashed, "\psi"] & & \ker(c) = \operatorname{im}(f) \arrow[u, "v"'] &
\end{tikzcd}$$

u is an epimorphism since it is a cokernel, and v is a monomorphism since it is a kernel. The existence of ψ comes from the fact that $f \circ k = 0$, so since u is a cokernel, by the definition of cokernel there is a unique map $\psi : \operatorname{coker}(k) \to B$. Note that $p \circ f = 0$, and since $f = \psi \circ u$ we have $p \circ \psi \circ u = 0$. Since u is an epimorphism, we have that, in fact, $p \circ \psi = 0$. Doing the same trick, this time thinking of kernels, since $p \circ \psi = 0$, we have a unique map $\sigma : \operatorname{coim}(f) \to \operatorname{im}(f)$. (I'll leave what's left of Exercise 2 (part 7) of the notes (Viterbo's notes, not these ones) for you to do.)

Exercise 2. *Show that the kernel of a monomorphism is (isomorphic to)* 0*, and the cokernel of an epimorphism is (isomorphic to)* 0*.*

Proposition 1. *Let $\mathcal{A}$ be an abelian category. Then a morphism which is both a monomorphism and an epimorphism is an isomorphism.*

Proof. Let $f : A \to B$ be a monomorphism and an epimorphism. By the previous exercise, the kernel/cokernel diagram for f is

$$0 \xrightarrow{k} A \xrightarrow{f} B \xrightarrow{c} 0$$

A moment's thought shows that the cokernel of k is the pair (A, id_A), and the kernel of c is the pair (B, id_B). We have the diagram

$$\begin{tikzcd}
0 \arrow[r, "k"] & A \arrow[r, "f"] \arrow[d, "\mathrm{id}_A"'] & B \arrow[r, "c"] & 0 \\
 & A = \operatorname{coim}(f) = \operatorname{coker}(k) \arrow[r, "\sigma"] & B = \ker(c) = \operatorname{im}(f) \arrow[u, "\mathrm{id}_B"'] &
\end{tikzcd}$$

Thus we have $f = \mathrm{id}_B \circ \sigma \circ \mathrm{id}_A$, and since all three on the right hand side are isomorphisms, so is f. $\square$

Now we move to the notion of an exact sequence.

Definition 13 (Exact Sequence)**.** *Let $\mathcal{A}$ be an abelian category. A sequence of maps $A \xrightarrow{f} B \xrightarrow{g} C$ in $\mathcal{A}$ is called* exact *if $g \circ f = 0$ and the natural map from $\operatorname{im}(f)$ to $\ker(g)$ is an isomorphism. We say the sequence is* split exact *if, in addition, there is a map $h : C \to B$ such that $g \circ h = id_C$.*

The map from $\operatorname{im}(f)$ to $\ker(g)$ is obtained via the following diagram:

$$\begin{tikzcd}
\operatorname{im}(f) \arrow[rr, dashed, "w"] \arrow[dr, "v"] & & \ker(g) \arrow[dl, "k"'] \\
A \arrow[u, "u"] \arrow[r, "f"] & B \arrow[r, "g"] & C
\end{tikzcd}$$

where u and v are the factorization of f from (2'). To obtain w notice that since $g \circ v \circ u = g \circ f = 0$ and u is an epimorphism, we have that $g \circ v = 0$, but then by definition of the kernel of g, we get a unique map $w : \operatorname{im}(f) \to \ker(g)$, as desired.

Note that $0 \to A \xrightarrow{f} B$ is exact iff f is a monomorphism, and $A \xrightarrow{f} B \to 0$ is exact iff f is an epimorphism. In fact,

Proposition 2. *If*

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is an exact sequence, then the kernel of g is the pair (A, f) and the cokernel of f is the pair (C, g).

Proof. Consider the diagram

$$\begin{tikzcd}
 & \operatorname{im}(f) \arrow[rr, "w"] \arrow[dr, "v"] & & \ker(g) \arrow[dl, "k"'] & \\
0 \arrow[r] & A \arrow[u, "u"] \arrow[r, "f"] & B \arrow[r, "g"] & C &
\end{tikzcd}$$

w is an isomorphism by assumption, so it will suffice to show that u is an isomorphism (again, u and v are as in (2')). Exercise 2 of the notes shows that u is an epimorphism, and since we have the factorization

$$\begin{tikzcd}
0 \arrow[r, "k"] & A \arrow[r, "u"] \arrow[rr, bend left, "f"] & \operatorname{im}(f) \arrow[r, "v"] & B
\end{tikzcd}$$

and f is a monomorphism, it follows that u is a monomorphism. Thus, since we are in an abelian category, u is an isomorphism, and hence the kernel of g is isomorphic to the pair (A, f). $\square$

Exercise 3.

(1) *Let $\mathcal{C}$ be a category and suppose that f is a morphism in $\mathcal{C}$ with $f = g \circ h$. If f is a monomorphism, show that h is a monomorphism. Likewise, if f is a epimorphism, show that g is an epimorphism.*
(2) *Prove the second claim in the proposition.*

Definition 14 ((Left/Right)-Exact Functor)**.** *Let F be a functor between additive categories. We say that F is* additive *if the associated map from $Hom\,(A, B)$ to $Hom\,(F(A), F(B))$ is an abelian group homomorphism. Let F be a functor between abelian categories. We say that F is* exact *if it transforms exact sequences into exact sequences. F is* left-exact *if it transforms an exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C$ into an exact sequence $0 \to F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C)$. It is called* right-exact *if it transforms a exact sequence $A \xrightarrow{f} B \xrightarrow{g} C \to 0$ into an exact sequence $F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C) \to 0$.*

We will close this section with some examples:

(1) Let $\mathcal{A}$ be an additive category and fix an object X in $\mathcal{A}$. The functor sending the object A to $\operatorname{Hom}_{\mathcal{A}}(X, A)$ (this is a functor from $\mathcal{A}$ to **Ab**) is left-exact.
(2) Fix a R-R-bimodule N. The functor from R-**Mod** to itself sending M to $N \otimes_R M$ is right-exact.

6.3. **The Category of Chain Complexes.** Given an abelian category $\mathcal{A}$, we may associate to it the category $\mathbf{Chain}(\mathcal{A})$ of *chain complexes in* $\mathcal{A}$. The objects of this category are sequences (often denoted $(A^\bullet, \partial)$):

$$\cdots \xrightarrow{d_{m-2}} A_{m-1} \xrightarrow{d_{m-1}} A_m \xrightarrow{d_m} A_{m+1} \xrightarrow{d_{m+1}} A_{m+2} \xrightarrow{d_{m+2}} \cdots$$

such that the *boundary maps* d_m satisfy $d_m \circ d_{m-1} = 0$. (Yes, these are *technically* cochain complexes, but Viterbo decides he wants to call them chain complexes, so rather than try to rewrite his notes, I will just call them chain complexes as well. We will still be taking the cohomology of them though.) The morphisms in this category are maps $u : A^\bullet \to B^\bullet$ such that

$$\begin{tikzcd}
\cdots \arrow[r, "d_{m-2}"] & A_{m-1} \arrow[r, "d_{m-1}"] \arrow[d, "u_{m-1}"] & A_m \arrow[r, "d_m"] \arrow[d, "u_m"] & A_{m+1} \arrow[r, "d_{m+1}"] \arrow[d, "u_{m+1}"] & A_{m+2} \arrow[r, "d_{m+2}"] \arrow[d, "u_{m+2}"] & \cdots \\
\cdots \arrow[r, "\partial_{m-2}"] & B_{m-1} \arrow[r, "\partial_{m-1}"] & B_m \arrow[r, "\partial_m"] & B_{m+1} \arrow[r, "\partial_{m+1}"] & B_{m+2} \arrow[r, "\partial_{m+2}"] & \cdots
\end{tikzcd}$$

is commutative.

This category has some natural subcategories, namely the subcategory of bounded complexes $\mathbf{Chain}^b(\mathcal{A})$, of complexes bounded below $\mathbf{Chain}^+(\mathcal{A})$, and complexes bounded above $\mathbf{Chain}^-(\mathcal{A})$. We may actually even think of $\mathcal{A}$ as a subcategory of $\mathbf{Chain}(\mathcal{A})$ by identifying A with $0 \to A \to 0$. This actually gives $\mathcal{A}$ as a full subcategory of $\mathbf{Chain}(\mathcal{A})$.

We can take the cohomology of a chain complex $(A^\bullet, d)$ as usual by defining the m^{th} cohomology to be $\mathscr{H}^m(A^\bullet) := \ker(d_m)/\operatorname{im}(d_{m-1})$. (The quotient is the cokernel of the natural map from $\operatorname{im}(d_{m-1})$ to $\ker(d_m)$.) We may actually consider the cohomology as a chain complex where the boundary maps are the zero map. We will denote this by $\mathscr{H}^*(A^\bullet)$.

We will now state a proposition without proof (or you can think of the proof as an exercise):

Proposition 3.

(1) *Let $\mathcal{A}$ be an abelian category. Then $\mathbf{Chain}^b(\mathcal{A})$, $\mathbf{Chain}^+(\mathcal{A})$, and $\mathbf{Chain}^-(\mathcal{A})$ are abelian categories.*
(2) *The map from $\mathbf{Chain}(\mathcal{A})$ to $\mathbf{Chain}(\mathcal{A})$ by taking cohomology is a functor. In particular, any morphism u from a complex $A^\bullet$ to $B^\bullet$ induces a map $u_* : \mathscr{H}^*(A^\bullet) \to \mathscr{H}^*(B^\bullet)$. If, moreover, u and v are* chain homotopic, *i.e. there is a map $P : A^\bullet \to B^\bullet$ such that $P_m : I_m \to J_{m-1}$ with $u - v = \partial_{m-1} \circ P_m + P_{m+1} \circ d_m$, then $u_* = v_*$.*

One last definition:

Definition 15 (Quasi-isomorphism)**.** *A map $u : A^\bullet \to B^\bullet$ is called a* quasi-isomorphism *if the induced map u_* is an isomorphism.*

A chain map $u : A^\bullet \to B^\bullet$ is a *chain homotopy equivalence* iff there is a chain map $v : B^\bullet \to A^\bullet$ such that $u \circ v$ and $v \circ u$ are chain homotopic to the appropriate identities. A chain homotopy equivalence is a quasi-isomorphism, but the converse is not true.

Proposition 4. *Given a short exact sequence of chain complexes*

$$0 \longrightarrow A^\bullet \longrightarrow B^\bullet \longrightarrow C^\bullet \longrightarrow 0$$

there is a long exact sequence

$$\cdots \to \mathscr{H}^m(A^\bullet) \to \mathscr{H}^m(B^\bullet) \to \mathscr{H}^m(C^\bullet) \xrightarrow{\delta} \mathscr{H}^{m+1}(A^\bullet) \to \cdots$$

Remark. *Suppose the original sequence in the proposition is split. Then we can construct a* sequence of chain maps

$$\cdots \to A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{g} C^\bullet \xrightarrow{\delta} A^\bullet[1] \xrightarrow{f[1]} B^\bullet[1] \to \cdots$$

where $(A^\bullet[k])_m = A_{k+m}$ *and* $d_{A^\bullet[k]} = (-1)^k d_{A^\bullet}$ *and the long exact sequence in the proposition is obtained by taking the cohomology of this sequence of chain maps.*

Finally, we will close on a big theorem that makes proving things in this category a lot easier, and an application of it:

Theorem 1 (Freyd-Mitchell)**.** *Let* $\mathcal{A}$ *be a small abelian category. There exists a ring* R *and a functor*

$$F : \mathcal{A} \to R - \mathbf{Mod}$$

which is fully faithful and exact.

As an example of an application of this theorem is

Lemma 1 (Snake Lemma)**.** *In an abelian category, consider a commutative diagram:*

$$\begin{tikzcd} & A \arrow[r, "f"] \arrow[d, "a"] & b \arrow[r, "g"] \arrow[d, "b"] & C \arrow[r] \arrow[d, "c"] & 0 \\ 0 \arrow[r] & A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' & \end{tikzcd}$$

where the rows are exact and 0 *is the zero object. Then there is an exact sequence relating the kernels and cokernels of* a*,* b*, and* c*:*

$$\ker(a) \to \ker(b) \to \ker(c) \xrightarrow{d} \mathit{coker}(a) \to \mathit{coker}(b) \to \mathit{coker}(c)$$

Further, if the morphism f *is a monomorphism, then so is* $\ker(a) \to \ker(b)$*, and if* g' *is an epimorphism, then so is* $\mathit{coker}(b) \to \mathit{coker}(c)$*.*

Sketch of Proof. Let's work in the abelian category generated by the objects and morphisms in the diagram. This is a small category, and so we may apply Freyd-Mitchell to think of all the objects as R-modules and morphisms as R-module homomorphisms. Since the functor given by Freyd-Mitchell is fully faithful, the map d constructed in R-**Mod** will have a corresponding map d in the original category. From here it is just commutative algebra. □

References

[1] T. Hungerford, *Algebra*, Springer GTM, (1974).
[2] S. Mac Lane, *Categories for the Working Mathematician, 2nd edition*, Springer GTM, (1998).
[3] C. Viterbo, *An Introduction to Symplectic Topology through Sheaf Theory*, http://www.math.polytechnique.fr/cmat/viterbo/Eilenberg/Eilenberg.pdf (2011).